

CAPABILITY OF NILPOTENT PRODUCTS OF CYCLIC GROUPS

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Abstract. A group is called capable if it is a central factor group. We consider the capability of nilpotent products of cyclic groups, and obtain a generalization of a theorem of Baer for the small class case. The approach is also used to obtain some recent results on the capability of certain nilpotent groups of class 2. We also prove a necessary condition for the capability of an arbitrary p -group of class k , and some further results.

1. INTRODUCTION

In his landmark paper on the classification of finite p -groups [11], P. Hall remarked that:

The question of what conditions a group G must fulfil in order that it may be the central quotient group of another group H , $G \cong H/Z(H)$, is an interesting one. But while it is easy to write down a number of necessary conditions, it is not so easy to be sure that they are sufficient.

Following M. Hall and Senior [10], we make the following definition:

Definition 1.1. A group G is said to be *capable* if and only if there exists a group H such that $G \cong H/Z(H)$; equivalently, if and only if G is isomorphic to the inner automorphism group of a group H .

Capability of groups was first studied by R. Baer in [3], where, as a corollary of some deeper investigations, he characterised the capable groups that are a direct sum of cyclic groups. Capability of groups has received renewed attention in recent years, thanks to results of Beyl, Felgner, and Schmid [5] characterising the capability of a group in terms of its epicenter; and more recently to work of Graham Ellis [6] that describes the epicenter in terms of the nonabelian tensor square of the group. The epicenter was used in [5] to characterize the capable extra-special p -groups; and the nonabelian tensor square was used in [1] to characterize the capable 2-generator finite p -groups of odd order and class 2.

While the nonabelian tensor product has proven very useful in the study of capable groups, it does seem to present certain limitations. At the end of [1], for example, the authors note that their methods require “very explicit knowledge of the groups” in question.

In the present work, we will use “low-tech” methods to obtain a number of results on capability of finite p -groups. They rely only on commutator calculus, and so may be more susceptible to extension than results that require explicit knowledge of the nonabelian tensor square of a group. In particular, we will prove a generalization of Baer’s Theorem characterising the capable direct sums of cyclic groups to the k -nilpotent products of cyclic p -groups with $k < p$, and with $k = p = 2$.

One weakness in our results should be noted explicitly: Baer’s Theorem fully characterizes the capable finitely generated abelian groups, because every finitely generated abelian group can be expressed as a direct sum of cyclic groups. Our generalization does not provide a result of similar reach, because whenever $k > 1$ there exist finite p -groups of class k that are not a k -nilpotent product of cyclic p -groups, even if we restrict to $p > k$. On the other hand, every finite p -group of class k is a *quotient* of a k -nilpotent product of cyclic p -groups, so some progress can be made from this starting point.

The paper is organized as follows: in Section 2 we present the basic definitions and notation. We proceed in Section 3 to establish a necessary condition for capability which extends an observation of P. Hall. In Section 4, we first describe the center of a k -nilpotent product of cyclic p -groups, where p is a prime satisfying $p \geq k$, and then use this description to prove Theorem 4.5, the promised generalization of Baer's Theorem. In Section 5 we characterize the capable 2-nilpotent products of cyclic 2-groups. Then in Section 6 we use our results on capable 2-nilpotent products of cyclic p -groups to derive a characterization of the capable 2-generated nilpotent p -groups of class two, p an odd prime (recently obtained through different methods by Bacon and Kappe), and some other related results, by way of illustration of how our methods may be used as a starting point for further investigations.

The main results are Theorem 3.19, giving a necessary condition for capability of a finite p group of class k , and Theorems 4.5 and 5.2, characterising the capable k -nilpotent products of cyclic p -groups for $p > k$ and for $k = p = 2$.

2. DEFINITIONS AND NOTATION

All maps will be assumed to be group homomorphisms. All groups will be written multiplicatively, unless we explicitly state otherwise.

Let G be a group. The center of G is denoted by $Z(G)$. The identity element of G will be denoted by e .

Let $x \in G$. We say that x is of *exponent* n if and only if $x^n = e$; we say x is of *order* $n > 0$ if and only if $x^n = e$ and $x^k \neq e$ for all k , $0 < k < n$.

The commutator $[x, y]$ of two elements x and y is defined to be $[x, y] = x^{-1}y^{-1}xy$; given two subsets (not necessarily subgroups) A and B of G , we let $[A, B]$ be the subgroup generated by all elements of the form $[a, b]$, with $a \in A$ and $b \in B$.

The lower central series of G is the sequence of subgroups defined by G_1, G_2, \dots , where $G_1 = G$, $G_{n+1} = [G_n, G]$. We say a group G is *nilpotent of class (at most) n* if and only if $G_{n+1} = \{e\}$; we will often drop the “at most” clause, it being understood. The nilpotent groups of class 1 are the abelian groups. Note that G is nilpotent of class n if and only if $G_n \subset Z(G)$. The class of all nilpotent groups of class at most k is denoted by \mathfrak{N}_k ; it is a variety of groups in the sense of General Algebra; we refer the reader to Hanna Neumann's excellent book [18].

We will write commutators left-normed, so that $[a_1, a_2, a_3] = [[a_1, a_2], a_3]$, etc.

The following properties of the lower central series are well-known. See for example [9]:

Definition 2.1. Let $g \in G$, $g \neq e$. We define $W(g)$ to be $W(g) = k$ if and only if $g \in G_k$ and $g \notin G_{k+1}$. We also set $W(e) = \infty$.

Proposition 2.2. Let G be a group.

- (i) For all $a, b \in G$, $W([a, b]) \geq W(a) + W(b)$.
- (ii) If $W(a_i) = w_1$ and $W(b_j) = w_2$, then

$$\left[\prod_{i=1}^I a_i^{\alpha_i}, \prod_{j=1}^J b_j^{\beta_j} \right] \equiv \prod_{i=1}^I \prod_{j=1}^J [a_i, b_j]^{\alpha_i \beta_j} \pmod{G_{w_1+w_2+1}}.$$

- (iii) If $a \equiv c \pmod{G_{W(a)+1}}$ and $b \equiv d \pmod{G_{W(b)+1}}$, then

$$[a, b] \equiv [c, d] \pmod{G_{W(a)+W(b)+1}}.$$

(iv) *A variant of the Jacobi identity:*

$$[a, b, c] [b, c, a] [c, a, b] \equiv e \pmod{G_{W(a)+W(b)+W(c)+1}}.$$

The following identities may be verified by direct calculation:

$$(2.3) \quad [xy, z] = [x, z][x, z, y][y, z]$$

$$(2.4) \quad [x, yz] = [x, z][z, [y, x]][x, y].$$

$$(2.5) \quad [a^r, b^s] \equiv [a, b]^{rs} [a, b, a]^{s \binom{r}{2}} [a, b, b]^{r \binom{s}{2}} \pmod{G_4}$$

$$(2.6) \quad [b^r, a^s] \equiv [a, b]^{-rs} [a, b, a]^{-r \binom{s}{2}} [a, b, b]^{-s \binom{r}{2}} \pmod{G_4},$$

where $\binom{r}{2} = \frac{r(r-1)}{2}$ for all integers r .

Nilpotent product of groups. The nilpotent products of groups were introduced by Golovin [8] as examples of regular products of groups. Although defined in a more general context in which there is no restriction on the groups involved, our definition will be restricted to the situation we are interested in.

Definition 2.7. Let $A_1, \dots, A_n \in \mathfrak{N}_k$. The k -nilpotent product of A_1, \dots, A_n , denoted by $A_1 \amalg^{\mathfrak{N}_k} \dots \amalg^{\mathfrak{N}_k} A_n$, is defined to be the group $G = F/F_{k+1}$, where F is the free product of the A_i , $F = A_1 * \dots * A_n$, and F_{k+1} is the $(k+1)$ -st term of the lower central series of F .

Note that the “1-nilpotent product” is simply the direct sum. Also, if the A_i are in \mathfrak{N}_{k-1} , and G is the k -nilpotent product of the A_i , then the $(k-1)$ -nilpotent product of the A_i is isomorphic to G/G_k .

The use of the coproduct notation does not appear to be standard in the literature, but there is a good reason to use it: the k -nilpotent product as defined above is the coproduct (in the sense of category theory) in the category \mathfrak{N}_k .

Basic commutators. The collection process of M. Hall gives normal forms for relatively free groups in \mathfrak{N}_k , and in some other special cases. The concept of basic commutators is essential in this development. The definition of basic commutators seems to vary in the literature (and sometimes even within the same work; cf. §11.1 and §12.3 in [9]). The definition we will use in the present work is that which appears in §12.3 of [9], where the ordering of commutators of weight $n > 1$ is given by letting $[x_1, y_1] < [x_2, y_2]$ if and only if $y_1 < y_2$ or $y_1 = y_2$ and $x_1 < x_2$ (lexicographically from right to left); here $[x_1, y_1]$ and $[x_2, y_2]$ are basic commutators of weight n .

For an exposition of the collection process, we direct the reader to Chapter 11 in [9]. The main consequences are Hall’s Basis Theorem [9, Theorem 11.2.4] and the collection formulas [9, §12.3]. We will use a slightly more precise version of the collection formulas in Section 3 below. We will also use later on a special case of the generalization of the Basis Theorem due to R.R. Struik [19, Theorem 3]; it states that if we take the k -nilpotent product of n cyclic p -groups, $p \geq k$, generated by x_1, \dots, x_n , then there is a normal form for the elements as products $\prod c_i^{a_i}$, where $c_1 < c_2 < \dots$ are the basic commutators on the x_i of weight less than or equal to k , and the exponent a_i is taken modulo the smallest of the orders of the generators that appear in the full expression of c_i .

A corollary of this result is that if F is the k -nilpotent product of the cyclic p -groups, $p \geq k$, then F_i/F_{i+1} is abelian with basis given by the basic commutators of weight exactly i , $i = 1, \dots, k$.

3. A NECESSARY CONDITION

In this section we give a necessary condition for capability of finite p -groups based on the orders of the elements on a minimal generating set. In the case of small class (that is, when G is a p -group with $G \in \mathfrak{N}_k$ and $p > k$), the condition reduces to an observation which goes back at least to P. Hall (penultimate paragraph in pp. 137 in [11]). Although Hall only considers bases in the sense of his theory of regular p -groups, his argument is essentially the same as the one we present. However, Hall's result may not be very well known, since it is only mentioned in passing; see for example [1, Theorem 4.4].

We begin by recalling three consequences of the collection process:

Lemma 3.1 (Lemma H1 in [19]). *Let x, y be any elements of a group; let c_1, c_2, \dots be the sequence of basic commutators of weight at least two in x and $[x, y]$, in ascending order. Then*

$$(3.2) \quad [x^\alpha, y] = [x, y]^\alpha c_1^{f_1(\alpha)} c_2^{f_2(\alpha)} \dots c_i^{f_i(\alpha)} \dots$$

where

$$(3.3) \quad f_i(\alpha) = a_1 \binom{\alpha}{1} + a_2 \binom{\alpha}{2} + \dots + a_{w_i} \binom{\alpha}{w_i},$$

where $a_i \in \mathbb{Z}$, and w_i is the weight of c_i in x and $[x, y]$. If the group is nilpotent, then the expression in (3.2) gives an identity, and the sequence of commutators terminates.

Lemma 3.4 (Lemma H2 in [19]). *Let α be a fixed integer, and G a nilpotent group of class at most k . If $b_j \in G$ and $r < k$, then*

$$(3.5) \quad [b_1, \dots, b_{i-1}, b_i^\alpha, b_{i+1}, \dots, b_r] = [b_1, \dots, b_r]^\alpha v_1^{f_1(\alpha)} v_2^{f_2(\alpha)} \dots v_t^{f_t(\alpha)}$$

where the v_k are (not necessarily basic) commutators in b_1, \dots, b_r of weight strictly greater than r , and every b_j , $1 \leq j \leq r$ appears in each commutator v_k . Furthermore, the f_i are of the form (3.3), with $a_j \in \mathbb{Z}$, and $w_i = w'_i - (r - 1)$, where w'_i is the weight of v_i in the b_i .

We find (3.5) useful in situations when we have commutators in some terms, some of which are shown as powers, and we want to “pull the exponent out.” At other times, we want to reverse the process and pull the exponents “into” a commutator. In such situations, we use (3.5) to express $[b_1, \dots, b_r]^\alpha$ in terms of other commutators. We will call the resulting identity (3.5'); that is:

$$(3.5') \quad [b_1, \dots, b_r]^\alpha = [b_1, \dots, b_{i-1}, b_i^\alpha, b_{i+1}, \dots, b_r] v_t^{-f_t(\alpha)} \dots v_2^{-f_2(\alpha)} v_1^{-f_1(\alpha)},$$

with the understanding that we will only do this in a nilpotent group so that the formula makes sense.

Lemma 3.6 (Theorem 12.3.1 in [9]). *Let x_1, \dots, x_s be any s elements of a group. Let c_1, c_2, \dots be the basic commutators in x_1, \dots, x_s of weight at least 2, written in increasing order. Then*

$$(3.7) \quad (x_1 \dots x_s)^\alpha = x_1^\alpha x_2^\alpha \dots x_s^\alpha c_1^{f_1(\alpha)} \dots c_i^{f_i(\alpha)} \dots$$

where $f_i(\alpha)$ is of the form (3.3), with a_j integers that depend only on c_i and not on α , and w_i is the weight of c_i in the x_j . If the group is nilpotent, then equation (3.7) is satisfied as an identity in the group, and the sequence of commutators terminates.

The following lemma is easily established by induction on the weight:

Lemma 3.8. *Let $F(y, z)$ be the free group on two generators. Then every basic commutator of weight ≥ 3 is of the form*

$$(3.9) \quad [z, y, y, c_4, \dots, c_r] \quad \text{or} \quad [z, y, z, c_4, \dots, c_r]$$

where $r \geq 3$, and c_4, \dots, c_r are basic commutators in y and z .

The main idea in our development is as follows: if we know that $[z, y^{p^i}]$ centralizes $\langle y, z \rangle$ in a group $G \in \mathfrak{N}_k$, for some prime p and all integers i greater than or equal to a given bound a , then we want to prove that a commutator of the form $[z^{p^n}, y]$ is equal to $[z, y^{p^n}]$. To accomplish this, we observe that a basic commutator of weight k will have exponent p^a , since we may simply use Lemma 3.8 and Proposition 2.2(ii) to pull the exponent into the second entry of the bracket. An arbitrary commutator of weight k will also have the same exponent, since G_k is abelian. For a basic commutator of weight $k - 1$ we may use (3.5') and deduce that a sufficiently high power of p will again yield the trivial element, by bounding below the power of p that divides the exponents $f_i(p^n)$. Then we apply Lemma 3.6 to deal with an arbitrary element of G_{k-1} . Continuing in this way, we can show that $\langle y, z \rangle_3$ is of exponent p^M for some M , and we will obtain the desired result by applying Lemma 3.1 to $[z^{p^N}, y]$ and $[z, y^{p^N}]$ for a suitably chosen large N , thus showing that they are both equal to $[z, y]^{p^N}$. Most of the work will go into obtaining a good estimate on how large this “large N ” has to be for everything to work.

If p is a prime and a is a positive integer, we let $[a]_p$ denote the exact p -divisor of a ; that is, we say that $[a]_p = r$ if $p^r | a$ and $p^{r+1} \nmid a$. Formally, we set $[0]_p = \infty$. A classical theorem of Kummer implies that if a is a positive integer, $0 < a \leq p^n$, then

$$\left[\binom{p^n}{a} \right]_p = n - [a]_p.$$

Recall that if x is a real number, then $\lfloor x \rfloor$ denotes the floor of x , the largest integer smaller than or equal to x .

Proposition 3.10. *Let p be a prime, n a positive integer, and m an integer with $0 < m \leq p^n$. If a_1, \dots, a_m are integers, then*

$$(3.11) \quad a_1 \binom{p^n}{1} + a_2 \binom{p^n}{2} + \dots + a_m \binom{p^n}{m} \equiv 0 \pmod{p^{n-d}},$$

where d is the smallest integer such that $p^{d+1} > m$; that is, $d = \lfloor \log_p(m) \rfloor$.

Proof. Write $m = m_0 + m_1 p + \dots + m_d p^d$, with $0 \leq m_i < p$ and $m_d > 0$. Then for all integers k between 1 and m , $0 \leq [k]_p \leq d$. Therefore, $n - [k]_p \geq n - d$, so each summand in (3.11) is divisible by p^{n-d} , as claimed. \square

Lemma 3.12. *Let $k \geq 3$, and let $G \in \mathfrak{N}_k$. Let p be a prime, $a \geq 0$ an integer, and $y, z \in G$. Suppose that*

$$(3.13) \quad \forall i \geq a, \quad [z, y^{p^i}, y] = [z, y^{p^i}, z] = e.$$

Then $\langle y, z \rangle_k$ is of exponent p^a , and $\langle y, z \rangle_{k-1}$ is of exponent $p^{a + \lfloor \frac{1}{p-1} \rfloor}$.

Proof. It follows by (3.5'), (3.9), and our assumption that

$$[z, y, w, c_4, \dots]^{p^a} = [z, y^{p^a}, w, c_4, \dots] = e$$

where $w \in \langle y, z \rangle$, and $[z, y, w, c_4, \dots]$ is a basic commutator of weight k . Since $\langle y, z \rangle_k$ is abelian and generated by the basic commutators, this proves the first part of the statement.

Now let $c \in \langle y, z \rangle_{k-1}$ be a basic commutator of weight exactly $k-1$, and let $N \geq a$. Expressing c in the form given in (3.9), say $c = [z, y, w, c_r, \dots]$ with $w \in \langle y, z \rangle$, then (3.5') and our assumption yield

$$c^{p^N} = [z, y, w, c_4, \dots]^{p^N} = v_t^{-f_t(p^N)} \dots v_1^{-f_1(p^N)},$$

where all the v_i are of weight k . The corresponding exponents are of the form

$$f_i(p^N) = a_1 \binom{p^N}{1} + a_2 \binom{p^N}{2}.$$

By Proposition 3.10, this expression is divisible by $p^{N - \lfloor \log_p(2) \rfloor}$, and so whenever $N \geq a + \lfloor \log_p(2) \rfloor$, we have $c^{p^N} = e$. Therefore, every basic commutator of weight exactly $k-1$ is of exponent $p^{a + \lfloor \log_p(2) \rfloor}$. Since $k > 2$, G_{k-1} is abelian as well; as every generator is of exponent $p^{a + \lfloor \log_p(2) \rfloor}$, the theorem is now proven by noting that $\lfloor \log_p(2) \rfloor = \left\lfloor \frac{1}{p-1} \right\rfloor$. \square

We will need the following technical lemma:

Lemma 3.14. *Let k and n be positive integers, with $n > 1$. Then*

$$\max_{1 \leq s \leq k} \left(\left\lfloor \frac{k-s}{n-1} \right\rfloor + \lfloor \log_n(s+1) \rfloor \right) = \left\lfloor \frac{k}{n-1} \right\rfloor.$$

If $k \geq n-1$ then the maximum is always attained at $s = n-1$.

Proof. Fix k and n ; we may assume $k \geq n-1$. Let

$$f(s) = \frac{k-s}{n-1} + \lfloor \log_n(s+1) \rfloor, \quad s \in [1, k].$$

The function $f(s)$ is piecewise strictly decreasing on $[1, k]$, and continuous from the right at every point. So the maximum of $f(s)$ on $[1, k]$ will be achieved at $s = 1$, or at a point where $f(s)$ is not continuous. The points of discontinuity are the points of the form $s = n^r - 1$ where $1 \leq r \leq \lfloor \log_n(k+1) \rfloor$. At $s = 1$, we have

$$f(1) = \frac{k-1}{n-1} + \lfloor \log_n(2) \rfloor \leq \frac{k}{n-1},$$

with equality if and only if $n = 2$. At the points of discontinuity we have

$$\begin{aligned} f(n^r - 1) &= \frac{k}{n-1} - \frac{n^r - 1}{n-1} + r \\ &= \frac{k}{n-1} - (1 + n + n^2 + \dots + n^{r-1}) + r \leq \frac{k}{n-1}, \end{aligned}$$

with equality if and only if $r = 1$. Thus the maximum value of $f(s)$ is equal to $\frac{k}{n-1}$, always achieved at $s_0 = n-1$. The result now follows by taking $\lfloor f(s) \rfloor$. \square

Lemma 3.15 (cf. Lemma 8.83 in [16]). *Let G be a group, with $G \in \mathfrak{N}_k$ and $k \geq 3$. Furthermore, let p be a prime, $a > 0$ an integer, and $y, z \in G$. Assume that a, y, z , and G satisfy (3.13). Then $\langle y, z \rangle_{k-m}$ is of exponent $p^{a+\lfloor \frac{m}{p-1} \rfloor}$ for $m = 0, 1, \dots, (k-3)$.*

Proof. We proceed by induction on m . Lemma 3.12 proves cases $m = 0, 1$, so we let $m \geq 2$.

Assume the result is true for $j = 0, 1, \dots, m-1$. First, we consider a basic commutator c of weight exactly $k-m$. We may express c as in (3.9), and applying (3.5') to c^{p^N} we obtain

$$c^{p^N} = [z, y^{p^N}, w, c_4, \dots, c_r] v_t^{-f_t(p^N)} \dots v_2^{-f_2(p^N)} v_1^{-f_1(p^N)},$$

where $w \in \{y, z\}$, c_4, \dots, c_r are basic commutators in z and y , and the v_i are commutators in z, y, w, c_4, \dots, c_r ; we know that each of z, y, w, c_4, \dots, c_r appears at least once in each v_i , and that we may express the weight of v_i on z, y, w, c_4, \dots, c_r as $r+s$ for some positive integer s . Thus, we may conclude that if the weight of v_i in z, y, w, c_4, \dots, c_r is $r+s$, then $W(v_i) \geq (k-m) + s$. In particular, we consider only those commutators with $1 \leq s \leq m$.

If v_i is of weight $r+s$ in z, y, w, c_4, \dots, c_r , then by Lemma 3.4 we have that

$$f_i(p^N) = a_1 \binom{p^N}{1} + \dots + a_{s+1} \binom{p^N}{s+1}.$$

Therefore, $f_i(p^N)$ is divisible by $p^{N-\lfloor \log_p(s+1) \rfloor}$. Since $v_i \in \langle y, z \rangle_{(k-(m-s))}$, by the induction hypothesis we know that $v_i^{-f_i(p^N)}$ is trivial whenever $N - \lfloor \log_p(s+1) \rfloor \geq a + \left\lfloor \frac{m-s}{p-1} \right\rfloor$. Therefore, if

$$N \geq a + \left\lfloor \frac{m-s}{p-1} \right\rfloor + \lfloor \log_p(s+1) \rfloor$$

for $s = 1 \dots, m$, then we may conclude that $c^{p^N} = e$. By Lemma 3.14, the largest of these values is $a + \left\lfloor \frac{m}{p-1} \right\rfloor$, which shows that the basic commutators of weight exactly $m-k$ are of exponent $p^{a+\lfloor \frac{m}{p-1} \rfloor}$, as desired.

Now take an arbitrary element c of $\langle y, z \rangle_{(k-m)}$, and write $c = d_1 \dots d_r$, where $d_i = c_i^{\beta_i}$ is the power of a basic commutator of weight at least $k-m$ in y and z , and $c_1 < \dots < c_r$. To calculate c^{p^N} , we apply Lemma 3.6 to this expression. We obtain:

$$(d_1 \dots d_r)^{p^N} = c_1^{\beta_1 p^N} c_2^{\beta_2 p^N} \dots c_r^{\beta_r p^N} u_1^{f_1(p^N)} \dots u_i^{f_i(p^N)} \dots$$

where $f_i(p^N)$ is of the form (3.3), with w_i the weight of u_i in the d_j . If we let the weight of u_i in the d_j be equal to s , then we know that $2 \leq s$. Since $W(d_i) \geq (m-k)$, we have that $W(u_i) \geq s(m-k)$. In particular, we may restrict our attention to values of s satisfying $2 \leq s \leq \left\lfloor \frac{k}{k-m} \right\rfloor$.

If u_i is of weight s in the d_j , then it lies in $\langle y, z \rangle_{k-(m-(s-1)(k-m))}$, so by the induction hypothesis it is of exponent $p^{a+\lfloor \frac{m-(s-1)(k-m)}{p-1} \rfloor}$. By (3.3) and Proposition 3.10 we have that $f_i(p^N)$ is a multiple of $p^{N-\lfloor \log_p(s) \rfloor}$; thus we can guarantee that $u_i^{f_i(p^N)}$ is trivial if

$$N \geq a + \left\lfloor \frac{m-(s-1)(k-m)}{p-1} \right\rfloor + \lfloor \log_p(s) \rfloor.$$

Since $s \geq 2$ and $k - m \geq 3$, it is clear that this value will certainly be no larger than $a + \lfloor \frac{m}{p-1} \rfloor$ by Lemma 3.14, so we conclude that any element of $\langle y, z \rangle_{(k-m)}$ is of exponent $p^{a + \lfloor \frac{m}{p-1} \rfloor}$, as claimed. \square

Theorem 3.16 (cf. Corollary 8.84 in [16]). *Let $k \geq 1$ and $G \in \mathfrak{N}_{k+1}$. Furthermore, let p be a prime, $a > 0$ an integer, and y and z elements of G . Assume that*

$$\forall i \geq a, \quad [z, y^{p^i}, y] = [z, y^{p^i}, z] = e.$$

If $N \geq a + \lfloor \frac{k-1}{p-1} \rfloor$, then $[z^{p^N}, y] = [z, y]^{p^N} = [z, y^{p^N}]$.

Proof. Applying Lemma 3.1 to $[z^{p^N}, y]$ yields:

$$[z^{p^N}, y] = [z, y]^{p^N} v_1^{f_1(p^N)} v_2^{f_2(p^N)} \dots v_t^{f_t(p^N)},$$

where v_1, v_2, \dots are the basic commutators of weight at least 2 in z and $[z, y]$. If v_i is of weight $s \geq 2$ in z and $[z, y]$, then

$$f_i(p^N) = a_1 \binom{p^N}{1} + a_2 \binom{p^N}{2} + \dots + a_s \binom{p^N}{s},$$

and we know that $v_i \in \langle y, z \rangle_{s+1}$. By Theorem 3.15, $v_i^{f_i(p^N)}$ will be trivial if

$$N \geq a + \left\lfloor \frac{(k+1) - (s+1)}{p-1} \right\rfloor + \lfloor \log_p(s) \rfloor.$$

This must hold for $s = 2, \dots, k$. We set $s' = s - 1$ and rewrite the above as:

$$N \geq a + \left\lfloor \frac{(k-1) - s'}{p-1} \right\rfloor + \lfloor \log_p(s' + 1) \rfloor,$$

with $s' = 1, \dots, k-1$. By Lemma 3.14, the largest value the right hand side takes is $a + \lfloor \frac{k-1}{p-1} \rfloor$. This proves that $[z^{p^N}, y] = [z, y]^{p^N}$ if $N \geq a + \lfloor \frac{k-1}{p-1} \rfloor$. The proof that $[z, y^{p^N}] = [z, y]^{p^N}$ is essentially the same. \square

Remark 3.17. The results above also hold if the identity in (3.13) is replaced by an identity in which p^i is placed on any of the three entries in $[z, y, z]$, and in any of the three entries in $[z, y, y]$; for instance,

$$[z^{p^i}, y, z] = [z, y, y^{p^i}] = e, \quad \text{or} \quad [z, y^{p^i}, z] = [z^{p^i}, y, y] = e,$$

etc.

Lemma 3.18. *Let $k \geq 1$ be a positive integer, p a prime, and let H be a nilpotent p -group of class $k+1$. Suppose that y_1, \dots, y_r are elements of H such that their images generate $H/Z(H)$; assume further that the order of $y_i Z(H)$ in $H/Z(H)$ is p^{α_i} , with $1 \leq \alpha_1 \leq \dots \leq \alpha_r$. Then $\alpha_r \leq \alpha_{r-1} + \lfloor \frac{k-1}{p-1} \rfloor$; that is,*

$$y^{p^{\alpha_{r-1} + \lfloor \frac{k-1}{p-1} \rfloor}} \in Z(H).$$

Proof. Since H is generated by y_1, \dots, y_r and central elements, it is sufficient to prove that $[y_r^{p^{\alpha_{r-1} + \lfloor \frac{k-1}{p-1} \rfloor}}, y_j] = e$ for $j = 1, \dots, r-1$.

Note that $y_j^{p^{\alpha_{r-1}}}$ is central for $j = 1, \dots, r-1$, so that

$$\forall i \geq \alpha_{r-1}, \quad [y_r, y_j^{p^i}, y_j] = [y_r, y_j^{p^i}, y_r] = e.$$

Thus, we may apply Theorem 3.16 to conclude that

$$[y_r^{p^{\alpha_{r-1} + \lfloor \frac{k-1}{p-1} \rfloor}}, y_j] = [y_r, y_j^{p^{\alpha_{r-1} + \lfloor \frac{k-1}{p-1} \rfloor}}] = e,$$

proving our lemma. \square

The necessary condition is now immediate:

Theorem 3.19 (P. Hall if $k < p$ [11]). *Let G be a nilpotent p -group of class k , with $k \geq 1$ and p a prime. Furthermore, let $\{x_1, \dots, x_r\}$ be a generating set for G with x_i of order p^{α_i} , where $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_r$. If G is capable, then $r > 1$ and*

$$\alpha_r \leq \alpha_{r-1} + \left\lfloor \frac{k-1}{p-1} \right\rfloor.$$

Proof. Since a center-by-cyclic group is abelian, the necessity of $r > 1$ is clear. So assume that G is capable, and $r > 1$. Let H be a p -group of class $k+1$ such that $G \cong H/Z(H)$. Let y_1, \dots, y_r be elements of H that project onto x_1, \dots, x_r , respectively. Then Lemma 3.18 gives the condition on α_r , proving the theorem. \square

Remark 3.20. If G is of small class then $\lfloor \frac{k-1}{p-1} \rfloor = 0$, so Theorem 3.19 says that $\alpha_r \leq \alpha_{r-1}$; therefore, when $k < p$ the necessary condition becomes “ $r > 1$ and $\alpha_r = \alpha_{r-1}$.” This is Hall’s observation in [11], applied to a minimal generating set.

The obvious question to ask is whether the inequality is tight. It is clearly best possible if $k < p$. An easy case to consider for $k \geq p$ is $p = 2$. In this case, the dihedral group of order 2^{k+1} is of class k , minimally generated by an element of order 2 and an element of order 2^k ; and its central quotient is isomorphic to the dihedral group of order 2^k . Thus, the inequality is tight for all k when $p = 2$. The case of $k \geq p > 2$ is more difficult, but once again there is a 2-generator capable group where the equality holds, with one generator of order p . This is shown in [17]. These considerations lead to the following proposition.

Proposition 3.21 ([17]). *For every $k \geq 1$ and every prime p there exists a capable p -group of class k , minimally generated by an element of order p and an element of order $p^{1 + \lfloor \frac{k-1}{p-1} \rfloor}$. In particular, the bound in Theorem 3.19 is best possible.*

4. k -NILPOTENT PRODUCTS OF CYCLIC p -GROUPS, $k < p$

In this section we describe the center of a k -nilpotent product of cyclic p -groups in the case $k \leq p$; as a result of this description, we will prove the promised generalization of Baer’s Theorem for the small class case. The description is proven by induction on k , and we will need to do the case $k = 2$ explicitly.

Lemma 4.1. *Let p be a prime and C_1, \dots, C_r be cyclic p -groups generated by x_1, \dots, x_r , respectively, with p^{α_i} the order of x_i and $1 \leq \alpha_1 \leq \dots \leq \alpha_r$. If $G = C_1 \amalg^{\mathfrak{N}_2} \dots \amalg^{\mathfrak{N}_2} C_r$, then*

$$Z(G) = \langle x_r^{p^{\alpha_r-1}}, G_2 \rangle.$$

Proof. A theorem of T. MacHenry [14] shows that the cartesian $[B, A]$ of $A \amalg^{\mathfrak{N}_2} B$, with $A, B \in \mathfrak{N}_2$, is isomorphic to $B^{\text{ab}} \otimes A^{\text{ab}}$, the isomorphism being the one that sends $[b, a]$ to $\bar{b} \otimes \bar{a}$. Using the universal property of the coproduct, we map G

to $C_i \amalg^{n_2} C_r$ to conclude that $C_i \cap Z(G) = \{e\}$ for $i = 1, \dots, r-1$. Looking at $C_{r-1} \amalg^{n_2} C_r$ we also obtain that $C_r \cap Z(G) = \langle x_r^{p^{\alpha_{r-1}}} \rangle$, proving our claim. \square

We will also need two observations on basic commutators:

Lemma 4.2. *Let F be the free group on x_1, \dots, x_r . Let $[u, v]$ be a basic commutator in x_1, \dots, x_r , and assume that $\text{wt}([u, v]) = k \geq 2$.*

- (i) *If $v \leq x_r$, then $[u, v, x_r]$ is a basic commutator in x_1, \dots, x_r .*
- (ii) *If $v > x_r$, then $[u, v, x_r] \equiv [v, x_r, u]^{-1}[u, x_r, v] \pmod{F_{k+2}}$. In addition, both $[v, x_r, u]$ and $[u, x_r, v]$ are basic commutators in x_1, \dots, x_r .*

Proof. Clause (i) follows from the definition of basic commutator, as does the claim in clause (ii) that $[v, x_r, u]$ and $[u, x_r, v]$ are basic commutators. The congruence in (ii) follows from Proposition 2.2(iv). \square

Lemma 4.3. *Let F be the free group on x_1, \dots, x_r , and let $k \geq 2$. Let c_1, \dots, c_s be the basic commutators in x_1, \dots, x_r of weight exactly k listed in ascending order, and write $c_i = [u_i, v_i]$. Let β_1, \dots, β_s be any integers, and let $g = c_1^{\beta_1} \cdots c_s^{\beta_s}$. For $i = 1, \dots, s$, let d_i and f_i be defined by:*

$$\begin{aligned} d_i &= \begin{cases} [u_i, v_i, x_r] & \text{if } v_i \leq x_r; \\ [v_i, x_r, u_i]^{-1} & \text{if } v_i > x_r. \end{cases} \\ f_i &= \begin{cases} e & \text{if } v_i \leq x_r \\ [u_i, x_r, v_i] & \text{if } v_i > x_r. \end{cases} \end{aligned}$$

Then

$$[g, x_r] \equiv \prod_{i=1}^s d_i^{\beta_i} f_i^{\beta_i} \pmod{F_{k+2}}$$

and, except for removing the trivial terms with $f_i = e$, this expression is in normal form for the abelian group F_{k+1}/F_{k+2} .

Proof. It is easy to verify that if $[u_i, v_i] \neq [u_j, v_j]$, then d_i, f_i, d_j, f_j will be pairwise distinct, except perhaps in the case where $f_i = f_j = e$. Thus in the expression given for $[g, x_r]$, all terms are either powers of the identity or of pairwise distinct basic commutators. That the expression is indeed equal to $[g, x_r]$ follows from Proposition 2.2(ii). \square

We are now ready to prove our result:

Theorem 4.4. *For a positive integer k and a prime p with $p \geq k$, let C_1, \dots, C_r be cyclic p -groups generated by x_1, \dots, x_r respectively, with p^{α_i} being the order of x_i , and assume that $1 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_r$. If G is the k -nilpotent product of the C_i , $G = C_1 \amalg^{n_k} \cdots \amalg^{n_k} C_r$, then $Z(G) = \langle x_r^{p^{\alpha_{r-1}}}, G_k \rangle$.*

Proof. That the center contains the right hand side follows from Lemma 3.18 and the properties of a k -nilpotent product. To prove the other inclusion, we proceed by induction on k . The result is trivially true for $k = 1$, and Lemma 4.1 gives the result for $k = 2$.

Assume the result is true for the $(k-1)$ -nilpotent product of the C_i , $2 < k \leq p$, and let $K = C_1 \amalg^{n_{k-1}} \cdots \amalg^{n_{k-1}} C_r$; that is, $K = G/G_k$.

By the induction hypothesis, we know that $Z(K) = \langle x_r^{p^{\alpha_{r-1}}}, K_{k-1} \rangle$, and the center of G is contained in the pullback of this subgroup. So we have the inclusions $\langle x_r^{p^{\alpha_{r-1}}}, G_k \rangle \subseteq Z(G) \subseteq \langle x_r^{p^{\alpha_{r-1}}}, G_{k-1} \rangle$. Let $g \in Z(G)$; multiplying by adequate

elements of G_k and an adequate power of $x_r^{p^{\alpha_r-1}}$, we may assume that g is an element of G_{k-1} which can be written in normal form as:

$$g = \prod_{i=1}^n c_i^{\beta_i},$$

where c_1, \dots, c_n are the basic commutators of weight exactly $k-1$ in x_1, \dots, x_r , and the β_i are integers on the relevant interval. If we prove that $g = e$, we will obtain our result.

Since $g \in Z(G)$, its commutator with x_r is trivial. From Proposition 2.2(ii) we have:

$$e = [g, x_r] = \left[\prod_{i=1}^n c_i^{\beta_i}, x_r \right] = \prod_{i=1}^n [c_i, x_r]^{\beta_i}.$$

For each $i = 1, \dots, n$, write $c_i = [u_i, v_i]$, with u_i, v_i basic commutators. Let d_i and f_i be as in the statement of Lemma 4.3. Then we have

$$e = [g, x_r] = \prod_{i=\ell}^n d_i^{\beta_i} f_i^{\beta_i}.$$

Except for some f_i which are trivial, the precise ordering of the remaining terms, and the exponents for the d_i corresponding to $v_i > x_r$, this is already in normal form. The ordering of the nontrivial basic commutators is immaterial, since the d_i and f_j commute pairwise; and for those d_i corresponding to $v_i > x_r$, we simply add the corresponding power of p to the exponent $-\beta_i$ to obtain an exponent in the correct range (see [19, Theorem 3]). The expression is then in normal form, so the only way in which this product can be the trivial element is if $\beta_i = 0$ for $i = 1, \dots, n$, proving that $g = e$, and so the theorem. \square

We now have the desired generalization:

Theorem 4.5 (Baer [3] for $k = 1$). *For k a positive integer and p a prime with $p > k$, let C_1, \dots, C_r be cyclic p -groups generated by x_1, \dots, x_r respectively, where p^{α_i} is the order of x_i with $1 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_r$. If G is the k -nilpotent product of the C_i , $G = C_1 \amalg^{\mathfrak{N}_k} \dots \amalg^{\mathfrak{N}_k} C_r$, then G is capable if and only if $r > 1$ and $\alpha_{r-1} = \alpha_r$.*

Proof. Necessity follows from Theorem 3.19. For sufficiency, let

$$K = C_1 \amalg^{\mathfrak{N}_{k+1}} \dots \amalg^{\mathfrak{N}_{k+1}} C_r.$$

Since $\alpha_{r-1} = \alpha_r$, we have from Theorem 4.4 that $Z(K) = K_{k+1}$, so $K/Z(K) = K/K_{k+1} \cong G$, as desired. \square

Remark 4.6. The original statement of Baer's Theorem is not restricted to torsion groups, or even to finitely generated groups. The original result is:

Baer's Theorem (Corollary to Existence Theorem in [3]). *A direct sum G of cyclic groups (written additively) is capable if and only if it satisfies the following two conditions:*

- (i) *If the rank of G/G_{tor} is 1, then the orders of the elements in G_{tor} are not bounded; and*
- (ii) *If $G = G_{\text{tor}}$, and the rank of $(p^{i-1}G)_p / (p^iG)_p$ is 1, then G contains elements of order p^{i+1} , for all primes p ;*

where $kG = \{kx \mid x \in G\}$, and $H_p = \{h \in H \mid ph = 0\}$ for any subgroup H of G .

The reader may find it interesting to give a proof of Baer's Theorem using our methods and the 2-nilpotent product; a complete description of the multiplication table for the 2-nilpotent product is given by Golovin in [7]. The proof of Baer's Theorem is then straightforward, and only complicated by the notation needed to consider infinite direct sums.

5. THE CASE $k = p = 2$

In this section we consider the smallest case that is not covered by our investigations so far: $k = p = 2$. In this instance, Theorem 3.19 gives the condition $\alpha_r \leq \alpha_{r-1} + 1$. We will prove that the condition is also sufficient for the case of the 2-nilpotent product of cyclic 2-groups.

As before, we start by examining the center of a 3-nilpotent product of cyclic 2-groups. Such a product was considered in detail by R.R. Struik in [19, 20]. To obtain uniqueness in the normal form, we must replace the basic commutators $[z, y, z]$ and $[z, y, y]$ with commutators $[z^2, y]$ and $[z, y^2]$, respectively, and adjust the ranges of the exponents accordingly. The normal form result we obtain with these changes is described in [19, Theorem 4]; explicitly, it states that if C_1, \dots, C_r are cyclic groups generated by x_1, \dots, x_r respectively, and if the order of x_i is 2^{α_i} , $1 \leq \alpha_1 \leq \dots \leq \alpha_r$, then every element of the 3-nilpotent product of the C_r may be written uniquely in normal form as:

$$g = x_1^{\gamma_1} \cdots x_r^{\gamma_r} \prod_{1 \leq i < j \leq r} [x_j, x_i]^{\gamma_{ji}} [x_j^2, x_i]^{\gamma_{jij}} [x_j, x_i^2]^{\gamma_{jii}} \prod_{1 \leq i < j < k \leq r} [x_j, x_i, x_k]^{\gamma_{jik}} [x_k, x_i, x_j]^{\gamma_{kij}},$$

where γ_i, γ_{jik} , and γ_{kij} are integers modulo 2^{α_i} ; γ_{jii} is an integer modulo 2^{α_i+1} ; γ_{jij} is an integer modulo 2^{α_i-1} ; and γ_{jij} is an integer modulo 2^{α_i-1} if $\alpha_i = \alpha_j$, and modulo 2^{α_i} if $\alpha_i < \alpha_j$.

Struik also provides multiplication formulas in [19, pp. 453], which may be verified by applying the collection process; note however that our choice of basic commutators differs slightly from hers, so the formulas in [19] do not apply as is to our choice of normal forms. It is straightforward to do the necessary conversions. In any case we will not need the multiplication formulas here.

To simplify notation we set $\alpha = \alpha_{r-1}$. Let G be as above; we want to determine the center of G . Clearly, $G_3 \subset Z(G)$, and from Lemma 3.18 we know that $x_r^{2^{\alpha+1}}$ is central. By considering G/G_3 , we obtain the 2-nilpotent product of the C_i , and so we know from Lemma 4.1 that the center of G satisfies:

$$\langle x_r^{2^{\alpha+1}}, G_3 \rangle \subset Z(G) \subset \langle x_r^{2^\alpha}, G_2 \rangle.$$

We claim that in fact

$$\langle x_r^{2^{\alpha+1}}, G_3 \rangle \subset Z(G) \subset \langle x_r^{2^{\alpha+1}}, G_2 \rangle.$$

This can be seen as follows: if $\alpha = \alpha_r$, then $x_r^{2^{\alpha+1}} = x_r^{2^\alpha} = e$, and the claim is true. If, on the other hand, we have $\alpha < \alpha_r$, then consider the commutator of x_{r-1} with $x_r^{2^\alpha}$. From (2.5) we have:

$$\begin{aligned} [x_r^{2^\alpha}, x_{r-1}] &= [x_r, x_{r-1}]^{2^\alpha} [x_r, x_{r-1}, x_r]^{\binom{2^\alpha}{2}} \\ &= [x_r, x_{r-1}]^{2^\alpha - 2^\alpha} [x_r^2, x_{r-1}]^{2^{\alpha-1}} \\ &= [x_r^2, x_{r-1}]^{2^{\alpha-1}}; \end{aligned}$$

where we have used the identity $[x_j, x_i, x_j] = [x_j, x_i]^{-2}[x_j^2, x_i]$, which may be obtained from (2.5) as well. Since $\alpha = \alpha_{r-1} < \alpha_r$, the order of $[x_r^2, x_{r-1}]$ is 2^α , so this commutator is not trivial. Therefore, $x_r^{2^{\alpha_{r-1}}}$ is not central, but its square is, giving once again the inclusions claimed.

Now let $g \in Z(G)$; multiplying by suitable powers of x_r and of commutators of the form $[x_j, x_i, x_k]$ and $[x_k, x_i, x_j]$ we may assume that:

$$g = \prod_{1 \leq i < j \leq r} [x_j, x_i]^{\gamma_{ji}} [x_j^2, x_i]^{\gamma_{jij}} [x_j, x_i^2]^{\gamma_{jii}}.$$

We first take the commutator of g with x_r ; from (2.3) and Proposition 2.2(ii) we have:

$$\begin{aligned} [g, x_r] &= \prod_{1 \leq i < j \leq r} [x_j, x_i, x_r]^{\gamma_{ji}} [x_j^2, x_i, x_r]^{\gamma_{jij}} [x_j, x_i^2, x_r]^{\gamma_{jii}} \\ &= \prod_{1 \leq i < j \leq r} [x_j, x_i, x_r]^{\gamma_{ji} + 2\gamma_{jij} + 2\gamma_{jii}}. \end{aligned}$$

For $j < r$, we conclude that $\gamma_{ji} + 2\gamma_{jij} + 2\gamma_{jii} \equiv 0 \pmod{2^{\alpha_i}}$. If $j = r$ we again use the identity $[x_r, x_i, x_r] = [x_r, x_i]^{-2}[x_r^2, x_i]$ and conclude that we must have $-2(\gamma_{ri} + 2\gamma_{rir} + 2\gamma_{rii}) \equiv 0 \pmod{2^{\alpha_{i+1}}}$ by looking at the exponent of $[x_r, x_i]$ in the resulting expression. So in any case we conclude that

$$\gamma_{ji} + 2\gamma_{jij} + 2\gamma_{jii} \equiv 0 \pmod{2^{\alpha_i}}; \quad 1 \leq i < j \leq r.$$

Conversely, it is easy to verify that if g as above satisfies this condition, then it will necessarily be central in G . We obtain:

Theorem 5.1. *Let C_1, \dots, C_r be cyclic groups, generated by x_1, \dots, x_r respectively; assume that the order of x_i is 2^{α_i} , and that $1 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_r$. Let G be the 3-nilpotent product of the C_i , $G = C_1 \amalg^{\mathfrak{N}_3} \dots \amalg^{\mathfrak{N}_3} C_r$, and write $\alpha = \alpha_{r-1}$. Then $g \in Z(G)$ if and only if it can be written in normal form as:*

$$g = x_r^{\gamma_r} \prod_{1 \leq i < j \leq r} [x_j, x_i]^{\gamma_{ji}} [x_j, x_i^2]^{\gamma_{jii}} [x_j^2, x_i]^{\gamma_{jij}} \prod_{1 \leq i < j < k \leq r} [x_j, x_i, x_k]^{\gamma_{jik}} [x_k, x_i, x_j]^{\gamma_{kij}},$$

where $\gamma_r \equiv 0 \pmod{2^{\alpha+1}}$, and $\rho_{ji} \equiv 0 \pmod{2^{\alpha_i}}$, where $\rho_{ji} = \gamma_{ji} + 2\gamma_{jii} + 2\gamma_{jij}$. That is

$$Z(G) = \left\langle \left\{ x_r^{2^{\alpha+1}}, G_3 \right\} \cup \left\{ [x_j, x_i]^{2^{\alpha_i}} \mid 1 \leq i < j \leq r \right\} \right\rangle.$$

With a description of the center, we can now easily derive the characterization of the capable 2-nilpotent products of cyclic 2-groups:

Theorem 5.2. *Let C_1, \dots, C_r be cyclic 2-groups generated by x_1, \dots, x_r , respectively, where 2^{α_i} is the order of x_i , and assume that $1 \leq \alpha_1 \leq \dots \leq \alpha_r$. If G is the 2-nilpotent product of the C_i ,*

$$G = C_1 \amalg^{\mathfrak{N}_2} C_2 \amalg^{\mathfrak{N}_2} \dots \amalg^{\mathfrak{N}_2} C_r,$$

then G is capable if and only if $r > 1$ and $\alpha_r \leq \alpha_{r-1} + 1$.

Proof. Necessity follows from Theorem 3.19. For sufficiency, let K be the 3-nilpotent product of the C_i , $K = C_1 \amalg^{\mathfrak{N}_3} C_2 \amalg^{\mathfrak{N}_3} \dots \amalg^{\mathfrak{N}_3} C_r$. Then the description of the center at the end of Theorem 5.1 makes it easy to verify that $K/Z(K) \cong G$, so G is capable. \square

The preceding theorem suggests the following question:

Question 5.3. *Is the k -nilpotent product of r cyclic p -groups capable if and only if $r > 1$ and*

$$\alpha_r \leq \alpha_{r-1} + \left\lfloor \frac{k-1}{p-1} \right\rfloor ?$$

As we have seen, the answer is yes when $k < p$, and when $k = p = 2$.

6. SOME APPLICATIONS OF OUR APPROACH

As we noted in the introduction, one weakness of Theorems 4.5 and 5.2 is that whereas the 1-nilpotent product of cyclic groups covers all finitely generated abelian groups, the case $k \geq 2$ does not do the same for the finitely generated nilpotent groups of class k . However, it is possible to use our results as a starting point for discussing capability of other more general p -groups.

A recent result of Bacon and Kappe characterizes the capable 2-generated nilpotent p -groups of class two with p an odd prime using the nonabelian tensor square [1]. We can recover their result using our techniques, and easily obtain a bit more.

In Theorem 2.4 of [2], the authors present a classification of the finite 2-generator p -groups of class two, p an odd prime. With a view towards their calculations of the nonabelian tensor square, the authors classify the groups into three families. We will modify their classification and coalesce them into a single presentation.

Let $G = \langle a, b \rangle$ be a finite nonabelian 2-generator p -group of class 2, p an odd prime. Then G is isomorphic to the group presented by:

$$(6.1) \quad \left\langle a, b \mid \begin{array}{lcl} a^{p^\alpha} = b^{p^\beta} = [b, a]^{p^\gamma} & = & e, \\ [a, b, a] = [a, b, b] & = & e, \\ a^{p^{\alpha+\sigma-\gamma}} [b, a]^{p^\sigma} & = & e. \end{array} \right\rangle$$

where $\alpha + \sigma \geq 2\gamma$, $\beta \geq \gamma \geq 1$, $\gamma \geq \sigma \geq 0$, and if $\sigma = \gamma$, then $\alpha \geq \beta$. Under these restrictions, the choice is uniquely determined.

If $\sigma = \gamma$, we obtain the groups in Bacon and Kappe's first family, which one might call the "coproduct type" groups (they are obtained from the nilpotent product $\langle a \rangle \amalg^{\mathfrak{N}_2} \langle b \rangle$ by taking the quotient modulo a power of $[a, b]$, and are the coproduct in a suitably chosen subvariety of \mathfrak{N}_2). If $\sigma = 0$, we obtain the split meta-cyclic groups, which are the second family in [2]. The cases $0 < \sigma < \gamma$ correspond to their third family.

The result which appears in [1] is that a 2-generated group with presentation as in (6.1) with $\sigma = 0$ or $\sigma = \gamma$ is capable if and only if $\alpha = \beta$. The condition is also both necessary and sufficient for the remaining case with $0 < \sigma < \gamma$ (in this case, [1] contains an error which the authors are in the process of correcting [13]).

Thus, in the case of 2-generated p -groups of class two, p an odd prime, Baer's condition is both necessary and sufficient, just as for finite abelian groups.

Although we could prove the result for all three families at once, we will divide the argument in two, to prove slightly more for the case where $\sigma = \gamma$.

Let p be an odd prime and C_1, \dots, C_r be cyclic p -groups generated by x_1, \dots, x_r , respectively, where p^{α_i} is the order of x_i with $1 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_r$. Let K be the 2-nilpotent product of the C_i ,

$$K = C_1 \amalg^{\mathfrak{N}_2} \dots \amalg^{\mathfrak{N}_2} C_r.$$

For each pair j, i , let β_{ji} be a positive integer less than or equal to α_i , and set $N = \langle [x_j, x_i]^{p^{\beta_{ji}}} \rangle$. Since N is central, we have $N \triangleleft K$. Let $G = K/N$.

Theorem 6.2. *The group G , as defined in the previous paragraph, is capable if and only if $r > 1$ and $\alpha_{r-1} = \alpha_r$.*

Proof. Necessity follows from Theorem 3.19. For sufficiency, let H be the 3-nilpotent product of the C_i , $H = C_1 \amalg^{\mathfrak{N}_3} \dots \amalg^{\mathfrak{N}_3} C_r$. Clearly H will not do, so we need to take a quotient of H so that, in the resulting group, $[x_j, x_i]^{\beta_{ji}}$ is central. To that end, we let M be the subgroup of H generated by all elements of the form $[x_j, x_i, x_k]^{\beta_{ji}}$ with $1 \leq i < j \leq r$, and k arbitrary. In terms of the basic commutators, this means the elements $[x_j, x_i, x_k]^{\beta_{ji}}$ for $k \geq i$, and the elements

$$[x_i, x_k, x_j]^{-\beta_{ji}} [x_j, x_k, x_i]^{\beta_{ji}}$$

for $k < i$. It is easy to verify that all elements of H/M have a normal form as in Theorem 3 of [19], and that the multiplication of these elements uses the same formulas as those in H , except that the exponents of the basic commutators of weight 3 are now taken modulo the adequate β_{ji} instead of the old moduli.

Proceeding now as in the proof of Theorem 4.4, one proves that the center of H/M is generated by the third term of the lower central series, the image of $x_r^{p^{\alpha_r-1}}$, and those of the elements $[x_j, x_i]^{p^{\beta_{ji}}}$; therefore by taking the central quotient, we obtain the group G , as desired. \square

The first part of [1, Corollary 4.3] corresponds to the case $r = 2$, $x_1 = a$, $x_2 = b$, and $\beta_{21} = \gamma$. Now we consider the case with $\sigma < \gamma$.

Theorem 6.3 (cf. [1, Corollary 4.3]). *Let p be an odd prime, and let G be a group presented by (6.1) with $0 \leq \sigma < \gamma$. Then G is capable if and only if $\alpha = \beta$.*

Proof. Necessity once again follows from Theorem 3.19; so we only need to prove sufficiency. The idea is to construct the “obvious witness” to the capability, by starting with the 3-nilpotent product of two cyclic groups of order p^α , generated by x and y ; then for every relation r in the presentation of G , we will take the quotient modulo the normal closure of $\langle [r, x], [r, y] \rangle$, thus making r central in the quotient. Then we just need to make sure that in the resulting group, the map $x \mapsto a$, $y \mapsto b$ will yield the desired isomorphism between the central quotient and G . In essence, what we are doing is constructing a “generalized extension of G ” which can be used to determine the capability of G . See [4, Theorem III.3.9].

So let $H = \langle x \rangle \amalg^{\mathfrak{N}_3} \langle y \rangle$, where x and y are both of order p^α .

First, we want to make sure that $[y, x]^{p^\gamma}$ is central. To that end, we let $N = \langle [y, x, x]^{p^\gamma}, [y, x, y]^{p^\gamma} \rangle$, and consider $K = H/N$.

The next step is to ensure that $x^{p^{\alpha+\sigma-\gamma}} [y, x]^{p^\sigma}$ is central. So first we consider

$$\begin{aligned} [x^{p^{\alpha+\sigma-\gamma}} [y, x]^{p^\sigma}, x] &= [x^{p^{\alpha+\sigma-\gamma}}, x] [x^{p^{\alpha+\sigma-\gamma}}, x, [y, x]^{p^\sigma}] [[y, x]^{p^\sigma}, x] \\ &= [y, x, x]^{p^\sigma} \end{aligned}$$

(using (2.3) and Proposition 2.2). So we let $L = K / \langle [y, x, x]^{p^\sigma} \rangle$.

So far, the only difference in the normal form and multiplication tables for L and for H is the order of $[y, x, x]$ and $[y, x, y]$.

The final quotient we need to take is to ensure that $x^{p^{\alpha+\sigma-\gamma}}[y, x]^{p^\sigma}$ also commutes with y . To that end, we consider:

$$\begin{aligned} [x^{p^{\alpha+\sigma-\gamma}}[y, x]^{p^\sigma}, y] &= [x^{p^{\alpha+\sigma-\gamma}}, y][x^{p^{\alpha+\sigma-\gamma}}, y, [y, x]^{p^\sigma}][[y, x]^{p^\sigma}, y] \\ &= [x^{p^{\alpha+\sigma-\gamma}}, y][y, x, y]^{p^\sigma} \\ &= [y, x]^{-p^{\alpha+\sigma-\gamma}}[y, x, x]^{-\binom{p^{\alpha+\sigma-\gamma}}{2}}[y, x, y]^{p^\sigma}. \end{aligned}$$

The last equality uses (2.5). Note, however, that the conditions on α , σ , and γ imply that $\alpha + \sigma - \gamma > \sigma$, so we can simplify the expression above to:

$$[x^{p^{\alpha+\sigma-\gamma}}[y, x]^{p^\sigma}, y] = [y, x]^{-p^{\alpha+\sigma-\gamma}}[y, x, y]^{p^\sigma}.$$

Let $N = \langle [y, x]^{p^{\alpha+\sigma-\gamma}}[y, x, y]^{-p^\sigma} \rangle$. It is easy to verify that for all $g \in L$, both $[g, x]$ and $[g, y]$ lie in N if and only if

$$g \in \left\langle x^{p^{\alpha+\sigma-\gamma}}[y, x]^{p^\sigma}, [y, x]^{p^\gamma}, [y, x, x], [y, x, y] \right\rangle,$$

(for example, using the explicit multiplication formulas in [19], suitably modified to fit our choice of basic commutators) from which we deduce that if $M = L/N$, then $M/Z(M) \cong G$, as desired. \square

Let p be an odd prime, and let G be a noncyclic 2-generator finite p -group of class at most two. If $G = \langle a, b \rangle$, then at least one of a and b must have the same exponent as the exponent of the element of highest order of the group, since G is regular; say a , with order p^μ . Among all elements $b' \in G$ such that $G = \langle a, b' \rangle$, pick the one of smallest order and replace b with b' ; say the order is p^ν . Then μ and ν are invariants of G ; in fact, they are the two largest type invariants of the group (see [12, §4]). Arguing as in [2, Lemma 2.3], one shows that $\langle a \rangle \cap \langle b' \rangle = \{e\}$, and then following the argument in [2, Th. 2.4], it is easy to verify that μ and ν are equal to α and β (in some order) in the presentation given in (6.1). So we obtain:

Corollary 6.4. *Let p be an odd prime, and let G be a noncyclic 2-generator finite p -group of class at most two, and let α and β be the two largest type invariants of G . Then G is capable if and only if $\alpha = \beta$.*

At this point, an obvious question to ask is whether the necessary condition of Theorem 3.19 will also prove to be necessary for the case of nilpotent groups of class two, as it was for abelian groups (at least, for odd prime and a suitable choice of minimal generating set). Unfortunately, the answer to that question is no.

Recall that a p -group G is *extra-special* if and only if $G' = Z(G)$, $|G'| = p$, and G^{ab} is of exponent p . The theorem of Beyl, Felgner, and Schmid in [5] mentioned in the introduction states that an extra-special p -group is capable if and only if it is dihedral of order order 8, or of order p^3 and exponent p , with $p > 2$. So the extra-special p -group G of order p^5 generated by x_1, x_2, x_3, x_4 , and satisfying $[x_3, x_1] = [x_3, x_2] = [x_4, x_1], [x_4, x_2] = [x_4, x_3] = [x_2, x_1] = e$, $x_1^p = x_2^p = x_3^p = x_4^p = e$, and $G_2 \subset Z(G)$, is not capable, and is minimally generated by four elements of exponent p . Thus, the necessary condition is not sufficient in general for groups in \mathfrak{N}_2 . An independent proof that G is not capable together with some other applications using our methods for groups of exponent p and class 2 appears in [15].

For now, we note that the nilpotent product can be used to produce a natural “candidate for witness” to the capability of a given finite nilpotent p -group G . Let G be a finite nilpotent group of class $k > 0$, minimally generated by elements

x_1, \dots, x_r . Let $w_1(x_1, \dots, x_r), \dots, w_n(x_1, \dots, x_r)$ be words in x_1, \dots, x_r that give a presentation for G , that is:

$$G = \langle x_1, \dots, x_r \mid w_1(x_1, \dots, x_r), \dots, w_n(x_1, \dots, x_r) \rangle.$$

Let $\langle y_1 \rangle, \dots, \langle y_r \rangle$ be infinite cyclic groups, let $K = \langle y_1 \rangle \amalg^{\mathfrak{N}_{k+1}} \dots \amalg^{\mathfrak{N}_{k+1}} \langle y_r \rangle$, and let $N = \langle w_1(y_1, \dots, y_r), \dots, w_n(y_1, \dots, y_r) \rangle^K$, that is, the normal closure of the subgroup generated by the words evaluated at y_1, \dots, y_r . Finally, let $M = [N, K]$.

Theorem 6.5. *The group G , as defined in the previous paragraph, is capable if and only if*

$$G \cong (K/M) / Z(K/M),$$

where K and M are also as defined in the previous paragraph.

Proof. We only need to prove the “only if” part. Note that the map that sends y_1, \dots, y_r to x_1, \dots, x_r respectively gives a well-defined map from K to G , and that $K/\langle N, K_{k+1} \rangle \cong G$; since $M \subseteq N$,

$$(K/M)/\langle NM, K_{k+1}M \rangle \cong G,$$

and $\langle NM, K_{k+1}M \rangle$ is central in K/M . Therefore, $(K/M)/Z(K/M)$ is a quotient of G . We want to show that it is actually isomorphic to G .

Since G is capable, there is a group H such that $H/Z(H) \cong G$. Let h_1, \dots, h_r be elements of H that map to x_1, \dots, x_r , respectively. Replacing H by $\langle h_1, \dots, h_r \rangle$ if necessary, we may assume that h_1, \dots, h_r generate H . Since G is of class k , H is of class $k+1$, and therefore there exists a unique surjective map from K (the relatively free \mathfrak{N}_{k+1} group of rank r) to H , mapping y_1, \dots, y_r to h_1, \dots, h_r , respectively. We must have that $w_i(h_1, \dots, h_r) \in Z(H)$ for $i = 1, \dots, n$, so the map from K to H factors through K/M . Since $Z(K/M)$ maps into $Z(H)$, the induced mapping $K/M \rightarrow H \rightarrow G$ factors through $(K/M)/Z(K/M)$. Therefore, we have that $(K/M)/Z(K/M)$ has G as a quotient. Since it is also isomorphic to a quotient of the finite group G , we must have $(K/M)/Z(K/M) \cong G$, as claimed. \square

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The last clause of Lemma 4.2(ii) as stated above is unfortunately false. For example, if the weight of $[v, x_r]$ is smaller than the weight of u , then $[v, x_r, u]$ cannot be a basic commutator; in addition, if u is of the form $[c_1, c_2]$ with c_i a basic commutator and the weight of c_2 is greater than 1, then $[u, x_r]$ is not a basic commutator, hence neither is $[u, x_r, v]$. Analogous problems arise if v is of this form.

The problem can be resolved by continuing the process, using work of Ward and of T.C. Hurley; a proof of Theorem 4.4 follows from those arguments. I am preparing a second part which will contain the correction of the problem, as well as extending the results to $k = p$ for arbitrary prime p .

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